LEVEL ONE REPRESENTATIONS OF $U_q(G_2^{(1)})$

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Abstract. We construct a level one representation of the quantum affine algebra $U_q(G_2^{(1)})$ by vertex operators from bosonic fields.

1. Introduction

Quantum affine algebras, the quantum groups associated to the affine Kac-Moody Lie algebras, provide an important underlying symmetry for the quantum Yang-Baxter equation [6] and quantum statistical models [11]. Explicit realizations of their representations are much needed in applications of quantum affine algebras. For instance, the Frenkel-Reshetikhin vertex operators [8] associated with the representations can be used to give solutions of the quantum Knizhnik-Zamoldchikov equation.

Lusztig first studied the abstract representations of quantum Kac-Moody algebras [20]. The program of constructing various representations was started in [7] for level one irreducible modules of ADE types, and subsequently twisted types were given in [12] and $B_n^{(1)}$ in [4]. Recently we have constructed symplectic quantum affine algebras in [17] for level one and in [16] for level $-1/2$. The case of $F_4^{(1)}$ can also be done similarly [15] using the idea of quantum Z-algebras [13, 15]. Besides the bosonic constructions, fermionic constructions were furnished in [10]. The q-Wakimoto construction was also known [21, 1, 22] afterwards. Other representations of classical quantum affine algebras have also been constructed [2]. The exceptional case of $G_2^{(1)}$ was the only case that has not been explicitly constructed.

The purpose of the paper is to give a explicit level one construction of the quantum affine algebra $U_q(G_2^{(1)})$ by vertex operators. The idea of the construction follows that of quantum Z-algebras [13, 15], which is a q-deformation of the classical $(q = 1)$ Z-algebras [19, 18]. We construct some auxiliary vertex operators for the short root. This is parallel to the known constructions of the affine Lie algebra $G_2^{(1)}$ [5, 9, 23], though the specialization of $q = 1$ in our construction is new even in the classical case.

The paper is organized as follows. In section two we review the quantum affine algebra $U_q(G_2^{(1)})$. Section three gives the Fock space representation of the quantum affine algebra $U_q(G_2^{(1)})$ stated in Theorem 3.1. Section four uses quantum vertex operator techniques to prove Theorem 3.1. In the proof of Serre relations we have to show a relation about certain symmetric functions, which is characteristic in the quantum affine algebras as noted in [12]. The Serre relations in $G_2^{(1)}$ turn out to

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be the most complicated one among both untwisted and twisted cases and actually capture all the existed phenomena in other types.

2. Quantum affine algebra $U_q(G_2^{(1)})$

Let α_i $(i = 1, 2)$ be the simple roots of the simple Lie algebra G_2, and λ_i be the fundamental weight. Let $P = \mathbf{Z}\varepsilon_1 + \mathbf{Z}\varepsilon_2$ and $Q = \mathbf{Z}\alpha_1 + \mathbf{Z}\alpha_2$ be the weight and root lattices. We then let $\Lambda_i, i \in I = \{0, 1, 2\}$ be the fundamental weights for the affine Lie algebra $G_2^{(1)}$, where $\Lambda_i = \lambda_i + \Lambda_0$, and λ_i are the fundamental weights for the finite dimensional simple Lie algebra G_2. The nondegenerate symmetric bilinear form $(\ |\ )$ on $\mathbf{h}^*$ is given by

$$(\alpha_i|\alpha_j) = d_i a_{ij}, \quad (\delta|\alpha_i) = (\delta|\delta) = 0 \ \text{ for all } i, j \tag{2.1}$$

where $(d_0, d_1, d_3) = (1, 1, 1/3)$ and $A = (a_{ij}) = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$.

Let $q_i = q^{d_i} = q^{\frac{1}{2}(\alpha_i|\alpha_i)}, i \in I$. The quantum affine algebra $U_q(G_2^{(1)})$ is the associative algebra with 1 over $\mathbf{C}(q^{1/6})$ generated by the elements $x_{ik}^{\pm}$, a_{il}, $K_i^{\pm 1}$, $\gamma^{\pm 1/2}$, $q^{\pm d}$ $(i = 1, 2, \cdots, n, k \in \mathbf{Z}, l \in \mathbf{Z} \setminus \{0\})$ with the following defining relations [6, 3, 14]:

$$[\gamma^{\pm 1/2}, u] = 0 \ \text{ for all } u \in \mathbf{U}, \tag{2.2}$$

$$[a_{ik}, a_{jl}] = \delta_{k+l,0} \frac{[(\alpha_i|\alpha_j)k]}{k} \frac{\gamma^k - \gamma^{-k}}{q - q^{-1}}, \tag{2.3}$$

$$[a_{ik}, K_j^{\pm 1}] = [q^{\pm d}, K_j^{\pm 1}] = 0, \tag{2.4}$$

$$q^d x_{ik}^{\pm} q^{-d} = q^k x_{ik}^{\pm}, \quad q^d a_{il} q^{-d} = q^l a_{il}, \tag{2.5}$$

$$K_i x_{jk}^{\pm} K_i^{-1} = q^{\pm(\alpha_i|\alpha_j)} x_{jk}^{\pm}, \tag{2.6}$$

$$[a_{ik}, x_{jl}^{\pm}] = \pm \frac{[(\alpha_i|\alpha_j)k]}{k} \gamma^{\mp|k|/2} x_{j,k+l}^{\pm}, \tag{2.7}$$

$$(z - q^{\pm(\alpha_i|\alpha_j)} w) X_i^{\pm}(z) X_j^{\pm}(w) + (w - q^{\pm(\alpha_i|\alpha_j)} z) X_j^{\pm}(w) X_i^{\pm}(z) = 0, \tag{2.8}$$

$$[X_i^+(z), X_j^-(w)] = \frac{\delta_{ij}}{q_i - q_i^{-1}} \left(\psi_i(w\gamma^{1/2}) \delta(\frac{w\gamma}{z}) - \varphi_i(w\gamma^{-1/2}) \delta(\frac{w\gamma^{-1}}{z}) \right) \tag{2.9}$$

where $X_i^{\pm}(z) = \sum_{n \in \mathbf{Z}} x_{i,n} z^{-n-1}$, ψ_{im} and φ_{im} $(m \in \mathbf{Z}_{\geq 0})$ are defined by

$$\sum_{m=0}^{\infty} \psi_{im} z^{-m} = K_i exp\left((q_i - q_i^{-1}) \textstyle\sum_{k=1}^{\infty} a_{ik} z^{-k} \right), \tag{2.10}$$

$$\sum_{m=0}^{\infty} \varphi_{i,-m} z^{m} = K_i^{-1} exp\left(-(q_i - q_i^{-1}) \textstyle\sum_{k=1}^{\infty} a_{i,-k} z^{k} \right), \tag{2.11}$$

$$\sum_{r=0, \sigma \in S_m}^{m=1-A_{ij}} (-1)^r \begin{bmatrix} m \\ r \end{bmatrix}_i \sigma . X_i^{\pm}(z_1) \cdots x_i^{\pm}(z_r) x_j^{\pm}(w) x_i^{\pm}(z_{r+1}) \cdots x_i^{\pm}(z_m) = 0. \tag{2.12}$$

where the symmetric group S_m acts on z_i by permuting their indices.

3. Fock space representations

Let $a_i(m)$ $(i = 1, 2)$ be the operators satisfying the Heisenberg relations for $U_q(G_2^{(1)})$ at $\gamma = q$ and $b(m)$ and $c(m)$ be two independent free bosonic operators with the relations:

$$[a_i(m), a_j(n)] = \delta_{m+l,0}\frac{[(\alpha_i|\alpha_j)m]}{m}[m] \tag{3.1}$$

$$[b(m), b(n)] = -\delta_{m+l,0}\frac{[2m/3]}{m}[m] \tag{3.2}$$

$$[c(m), c(n)] = \delta_{m+l,0}\frac{[2m/3]}{m}[5m/3] \tag{3.3}$$

$$[a_i(m), b(n)] = [a_i(m), c(n)] = [b(m), c(n)] = 0. \tag{3.4}$$

Let β_2 be an auxiliary simple root isomorphic to α_2. We define the Fock module $\mathcal{F}$ as the tensor product of the symmetric algebra generated by $a_i(-n), b(-n), c(-n)$ $(n \in \mathbb{N})$ and the twisted group algebra $\mathbb{C}\{\mathbb{P} + \mathbb{Z}\beta_{\not\models}\}$ generated by e^{α}, e^{β} subject to relation:

$$e^{\alpha_1}e^{\alpha_2} = -e^{\alpha_2}e^{\alpha_1}, \quad e^{\alpha}e^{\beta} = e^{\beta}e^{\alpha}, \quad e^{\alpha}e^{\alpha} = e^{\alpha}e^{\alpha}.$$

where $\alpha \in P$, and β is an element of the auxiliary lattice $\mathbb{Z}\alpha_{\not\models}$ (another copy of sublattice generated by the short root α_2). In the following we reserve β to denote an element from this auxiliary lattice.

The element 1 is the vacuum state. We define the action by

$$a_i(n).1 = 0 \quad (n > 0)\ , \quad b_i(n).1 = 0 \quad (n > 0)\ ,$$

The element $a_i(0), b(0)$ act as differential operators by

$$a_i(0)e^{\alpha} = (\alpha_i|\alpha)e^{\alpha}, \quad b(0)e^{\beta} = -\frac{2}{3}e^{\beta}.$$

As usual we define the normal product as the ordered product by moving annihilation operators $a_i(n), b(n), a_i(0), b(0)$ to the left.

Let's introduce the following operators.

$$Y_1^{\pm}(z) = \exp(\pm\sum_{n=1}^{\infty}\frac{a_1(-n)}{[n]}q^{\mp\frac{n}{2}}z^n)\exp(\mp\sum_{n=1}^{\infty}\frac{a_1(n)}{[n]}q^{\mp\frac{n}{2}}z^{-n})e^{\alpha_1}z^{\mp a_1(0)},$$

$$Y_2^{\pm}(z) = \exp(\pm\sum_{n=1}^{\infty}\frac{a_1(-n)+b(-n)}{[n]}q^{\mp\frac{n}{2}}z^n)\exp(\mp\sum_{n=1}^{\infty}\frac{a_2(n)+b(n)}{[n]}q^{\mp\frac{n}{2}}z^{-n}) \cdot$$
$$e^{\alpha+b}z^{\mp a_2(0\pm+b(0)},$$

$$U_{\pm}(z) = \exp(\mp\sum_{n=1}^{\infty}\frac{[n/3]}{[2n/3]}b(\pm n)z^{\mp n})q^{\mp b(0)/2},$$

$$W_{\pm}(z) = \exp(\mp\sum_{n=1}^{\infty}\frac{[n/3]}{[2n/3]}c(\pm n)z^{\mp n}).$$

Theorem 3.1. *The space $\mathcal{F}$ is a $U_q(G_2^{(1)})$-module of level one under the action defined by $\gamma \longmapsto q, K_i \longmapsto q^{a_i(0)}$, $a_{im} \longmapsto a_i(m), q^d \longmapsto q^{\overline{d}}$, and*

$$X_1^{\pm}(z) \longmapsto Y_1^{\pm}(z)$$
$$X_2^{\pm}(z) \longmapsto \frac{\pm Y_2^{\pm}(z)}{q_2 - q_2^{-1}} \left(U_{\pm}(q^{\mp 5/6}z) W_{\pm}(q^{\mp 1/2}z)^{\pm 1} - U_{\pm}(q^{\pm 5/6}z) W_{\pm}(q^{\pm 1/2}z)^{\mp 1} \right).$$

4. Proof of the Theorem

We now prove the theorem by checking that the action satisfy Drinfeld relations. It is clear that the relations (2.2-2.6) are true by the construction. The relation (2.7) follows from the definition of $Y_i^{\pm}(z)$ and the commutativity among $\alpha_i(n)$, $b(n)$ and $c(n)$. So we only need to show the relations (2.8-2.9).

We first compute the operator product expansions for $Y_i^{\pm}(z)$:

$$\begin{aligned} Y_i^{\pm}(z) Y_j^{\pm}(w) &=: Y_i^{\pm}(z) Y_j^{\pm}(w) : \\ &\quad \cdot exp(-\sum_{n=1}^{\infty} \frac{[(\alpha_i|\alpha_j)n]}{n[n]} q^{\mp n} (\frac{w}{z})^n) z^{(\alpha_i|\alpha_j)}. \\ Y_i^{\pm}(z) Y_j^{\mp}(w) &=: Y_i^{\pm}(z) Y_j^{\mp}(w) : \\ &\quad \cdot exp(\sum_{n=1}^{\infty} \frac{[(\alpha_i|\alpha_j)n]}{n[n]} (\frac{w}{z})^n) z^{-(\alpha_i|\alpha_j)}. \end{aligned} \tag{4.1}$$

For $\epsilon = \pm = \pm 1$ we define

$$\begin{aligned} X_{2\epsilon}^{+}(z) &= Y_2^{+}(z) U_{\epsilon}(q^{-5\epsilon/6}z) W_{\epsilon}(q^{-1\epsilon/2}z) \\ X_{2\epsilon}^{-}(z) &= Y_2^{-}(z) U_{\epsilon}(q^{5\epsilon/6}z) W_{\epsilon}(q^{1\epsilon/2}z)^{-1} \end{aligned}$$

so that $X_2^{\pm}(z) = \frac{1}{q_2 - q_2^{-1}} \left(X_{2+}^{\pm}(z) - X_{2-}^{\pm}(z) \right)$.

Note that for $i = j = 1$ the relation (2.8) follows from the $sl(2)$ case. For $(\alpha_i|\alpha_j) = -1$ (i.e. $i \neq j$) equation (4.1) becomes

$$Y_i^{\pm}(z) Y_j^{\pm}(w) =: Y_i^{\pm}(z) Y_j^{\pm}(w) : (z - q^{\mp 1}w)^{-1}, \tag{4.2}$$
$$Y_i^{\pm}(z) Y_j^{\mp}(w) =: Y_i^{\pm}(z) Y_j^{\mp}(w) : (z - q^{\mp 1}w), \tag{4.3}$$

which implies that for $i \neq j$

$$(z - q^{\mp 1}w) X_i^{\pm}(z) X_j^{\pm}(w) = (q^{\mp 1}z - w) X_j^{\pm}(w) X_i^{\pm}(z), \tag{4.4}$$
$$[X_1^{+}(z), X_2^{-}(w)] = 0, \tag{4.5}$$

where the latter is one case of relation (2.9).

To prove the remaining case of (2.8) we compute that

$$X_{2\epsilon}^{+}(z) X_{2\epsilon}^{+}(w) =: X_{2\epsilon}^{+}(z) X_{2\epsilon}^{+}(w) : \frac{z - w}{z - q^{2/3}w} q^{(1+\epsilon)/6} \tag{4.6}$$

Then we immediately get the "+" case of relation (2.8) for $i = j = 2$. The "-" case is shown similarly.

In relation (2.9), again we only need to show the cases involved with the short root α_2, since the proof of $[X_1^{+}(z), X_1^{-}(w)]$ is quite similar to that of type A [12]. Observe that

$$X_{2\epsilon}^{+}(z) X_{2,-\epsilon}^{-}(w) =: X_{2\epsilon}^{+}(z) X_{2,-\epsilon}^{-}(w) : . \tag{4.7}$$

Thus we reduce the relation to the commutators $[X^+_{2\epsilon}(z), X^-_{2\epsilon}(w)]$. We compute that

$$\begin{aligned}
&[X^+_{2+}(z), X^-_{2+}(w)]\\
&=: X^+_{2+}(z)X^-_{2+}(w) : (\frac{z-q^{5/3}w}{z-qw}q^{-1/3} - \frac{w-q^{-5/3}z}{w-q^{-1}z}q^{1/3})\\
&=: X^+_{2+}(z)X^-_{2+}(w) : \frac{z-q^{5/3}w}{z}q^{-1/3}\delta(\frac{qw}{z})\\
&= (q^{-1/3}-q^{1/3})\psi_2(zq^{1/2})\delta(\frac{qw}{z})
\end{aligned}$$

Similarly we can prove that

$$[X^+_{2-}(z), X^-_{2-}(w)] = (q^{1/3}-q^{-1/3})\phi_2(zq^{-1/2})\delta(\frac{q^{-1}w}{z}).$$

Finally we use the quantum vertex operator calculus [12] to prove the Serre relations. The case ($a_{12}=-1$) is similar to that of $U_q(A_n^{(1)})$ [12]. We only check the other one ($a_{ij}=-3$) in the "+" case:

$$\begin{aligned}
\sum_{\sigma\in S_4}\sigma.\big(&X_1^+(w)X_2^+(z_1)X_2^+(z_2)X_2^+(z_3)X_2(z_4)-\\
&[4]_2X_2^+(z_1)X_1^+(w)X_2^+(z_2)X_2^+(z_3)X_2^+(z_4)+\\
&\frac{[4]_2[3]_2}{[2]_2}X_2^+(z_1)X_2^+(z_2)X_1^+(w)X_2^+(z_3)X_2^+(z_4)-\\
&[4]_2X_2^+(z_1)X_2^+(z_2)X_2^+(z_3)X_1^+(w)X_2^+(z_4)+\\
&X_2^+(z_1)X_2^+(z_2)X_2^+(z_3)X_2(z_4)X_1^+(w)\big)=0
\end{aligned}\tag{4.8}$$

Recalling $X_2^{\pm}(z) = \frac{1}{q_2-q_2^{-1}}\left(X^{\pm}_{2+}(z)-X^{\pm}_{2-}(z)\right)$ and using (4.1-4.6) and Wick's theorem we can reduce the left-hand side of (4.8) into a linear combination of operator product terms $: X_1^+(w)X^+_{2\epsilon_1}(z_1)X^+_{2\epsilon_2}(z_2)X^+_{2\epsilon_3}(z_3)X_{2\epsilon_4}(z_4) :$, where $\epsilon_i=\pm$. Thus the Serre relation is equivalently reduced to four Serre-like relations grouped by the number of appearance of $X^+_{2+}(z_i)$ in the product. Due to (4.7) the most complicated contraction functions comes from the case when all $\epsilon_i=+$. All four subcases can be treated similarly. In the following we will only prove the case when $\epsilon_i=+$. Ignoring the factor $(q_2-q_2^{-1})^{-4}$ the left-hand side of this Serre-like relation is $q^2 : X_1^+(w)X^+_{2+}(z_1)X^+_{2+}(z_2)X^+_{2+}(z_3)X_{2+}(z_4) :$ times the following expression.

$$\begin{aligned}
&\sum_{\sigma\in S_4}\sigma.\prod_i\frac{1}{(w-q^{-1}z_i)(z_i-q^{-1}w)}\prod_{i<j}\frac{z_i-z_j}{z_i-q^{2/3}z_j}\cdot\\
&\quad\big[(z_1-q^{-1}w)\cdots(z_4-q^{-1}w)+[4]_2(w-q^{-1}z_1)(z_2-q^{-1}w)\cdots(z_4-q^{-1}w)+\\
&\quad\begin{bmatrix}4\\2\end{bmatrix}_2(w-q^{-1}z_1)(w-q^{-1}z_2)(z_3-q^{-1}w)(z_4-q^{-1}w)+\\
&\quad[4]_2(w-q^{-1}z_1)\cdots(w-q^{-1}z_3)(z_4-q^{-1}w)+(w-q^{-1}z_1)\cdots(w-q^{-1}z_4)\big]
\end{aligned}$$

where the symmetric group S_4 acts on the ring of rational functions in z_i by permutations on the indices. The q-binomial identity implies that the coefficients of 1

and w^4 are zero, and the expression in the bracket is then simplified to

$$q_2^4(q_2^{-6}-1)\left[q_2^{-1}w^3\left(q_2^{-12}z_1-q_2^{-6}(1+q_2^{-2}+q_2^{-4})z_2+q_2^{-2}(1+q_2^{-2}+q_2^{-4})z_3-z_4\right)\right.$$
$$+w^2(1+q_2^{-2})\left(q_2^{-12}z_1z_2-q_2^{-6}(1+q_2^{-2})z_1z_3+q_2^{-4}(1+q_2^{-2}+q_2^{-4})z_1z_4+\right.$$
$$\left.q_2^{-4}(1+q_2^{-2}+q_2^{-4})z_2z_3+q_2^{-4}(1+q_2^{-2})z_2z_4-z_3z_4\right)+$$
$$q_2^{-1}w\left(q_2^{-12}z_1z_2z_3-q_2^{-6}(1+q_2^{-2}+q_2^{-4})z_1z_2z_3+\right.$$
$$\left.\left.q_2^{-2}(1+q_2^{-2}+q_2^{-4})z_1z_3z_4-z_2z_3z_4\right)\right]$$

Let $f(z_1,z_2,z_3,z_4)$ denote the above expression. Since $\prod_{i<j}(z_i-q_2^2z_j)(z_i-q_2^{-2}z_j)$ is symmetric, we see that the Serre relation is equivalent to the following identity:

$$\sum_{\sigma\in S_4} sgn(\sigma)\sigma\left(f(z_1,z_2,z_3,z_4)\prod_{i<j}(z_i-q_2^{-2}z_j)\right)=0. \tag{4.9}$$

We claim that (4.9) is true. Notice that it is enough to check only the coefficients of w and w^2, and even the two are quite similar. The tedious checking of the coefficient of w shows that it is zero indeed. Thus the Serre relation is proved.

The constructed level one representation is reducible due to the presence of auxiliary bosons $b(m)$ and $c(m)$. All integrable irreducible level one modules are contained in the Fock representation and can be recovered by the technique of the screening operators.

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